Susy Extensions of Hopf maps

Francesco Toppan

CBPF (TEO), Rio de Janeiro (RJ), Brazil

Talk at New Trends in QG II, S. Paulo, 09/09/09.

Based on:
Gonzales-Rojas-FT 0812.3042 (to appear in IJMPA),
Gonzales-Kuznetsova-Nersessian-FT-Yeghikyan 0902.2682 (to appear in PRD),
Bellucci-FT-Yeghikyan 0905.3461,
Faria Carvalho-Kuznetsova-FT, in preparation,
Krivonos-Nersessian-FT, in preparation.

- minimal versus non-minimal linear multiplets.

- invariant actions.

- supersymmetric extension of the Schur lemma.

- bilinear embeddings (pre-Hopf maps)

- Non-linear supersymmetry induced by Hopf maps.

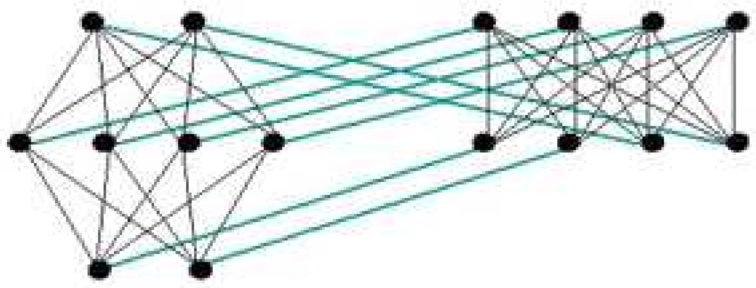
- $1D$ susy σ-models.

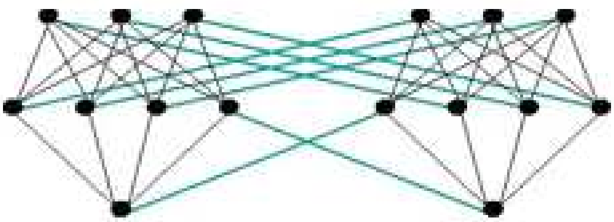
1D N-Extended Supersymmetry Algebra: N odd generators Q_i ($i = 1, \ldots, N$) and a single even generator H (the hamiltonian). It is defined by the (anti)-commutation relations

$$\begin{aligned}\{Q_i, Q_j\} &= 2\delta_{ij} H, \\ [Q_i, H] &= 0.\end{aligned}$$

$N = 5$ length-3 minimal linear supermultiplets:

fields cont.	$N = 4$ decomp.	ψ_g connectivities	labels
$(1,8,7)$	$(0,4,4)+(1,4,3)$	3_5+5_4	
$(2,8,6)$	$(0,4,4)+(2,4,2)$	$2_5+2_4+4_3$	A
	$(1,4,3)+(1,4,3)$	6_4+2_3	B
$(3,8,5)$	$(0,4,4)+(3,8,5)$	$1_5+3_4+4_2$	A
	$(1,4,3)+(2,4,2)$	$2_4+5_3+1_2$	B
$(4,8,4)$	$(0,4,4)+(4,4,0)$	4_4+4_1	A
	$(1,4,3)+(3,4,1)$	$1_4+3_3+3_2+1_1$	B
	$(2,4,2)+(2,4,2)$	4_3+4_2	C
$(5,8,3)$	$(1,4,3)+(4,4,0)$	$4_3+3_1+1_0$	A
	$(2,4,2)+(3,4,1)$	$1_3+5_2+2_1$	B
$(6,8,2)$	$(2,4,2)+(4,4,0)$	$4_2+2_1+2_0$	A
	$(3,4,1)+(3,4,1)$	2_2+6_1	B
$(7,8,1)$	$(3,4,1)+(4,4,0)$	5_1+3_0	





$N=5$ "oxidizes" to $N=8$ ($N=5\rightarrow N=8$) for minimal length-2 and 3 multiplets.

Manifestly $\overline{N}$-extended susy lagrangian $\mathcal{L}_{\overline{N}}$:

$$\mathcal{L}_{\overline{N}} \;=\; Q_1\cdots Q_{\overline{N}}F_{\overline{N}},$$

In mass-dimension, $[\mathcal{L}_{\overline{N}}]=2$, $[F_{\overline{N}}]=2-\frac{\overline{N}}{2}$.

N-extended supersymmetric action ($N>\overline{N}$) if$N-\overline{N}$ constraints,$j=\overline{N}+1,\cdots,N$:

$$Q_j\mathcal{L}_{\overline{N}} \;=\; \partial_t R_{j,\overline{N}},$$

with

$$[R_{j,\overline{N}}] \;=\; \frac{3}{2}.$$

$N=4$ invariant action: unconstrained prepotential F.

$N=5$ and beyond: Constrained prepotential F.

SO FAR: $N=5$ invariance $\rightarrow$ $N=8$ invariance.

The prepotential F induces a σ-model.

Example: the $N=5$ ($(2,8,6)_A$ or $(2,8,6)_B$) action implies the $N=8$ $(2,8,6)$ action.

Let $F(x,y)$ be the prepotential and $\Phi = \partial_x^2 F$. The $N=5$ constraint is

$$\partial_x^2\Phi + \partial_y^2\Phi \;=\; 0.$$

The $N=5$ ($N=8$) invariant lagrangian is

$$\begin{aligned}
\mathcal{L} \;=\;& \Phi(\dot{x}^2+\dot{y}^2-\psi_0\dot{\psi}_0-\psi_i\dot{\psi}_i-\lambda_0\dot{\lambda}_0-\lambda_i\dot{\lambda}_i+g_ig_i+f_if_i)+\\
&+\Phi_x[\dot{y}(\psi_0\lambda_0-\psi_i\lambda_i)-g_i(\psi_i\psi_0+\lambda_i\lambda_0)+f_i(\psi_i\lambda_0-\lambda_i\psi_0)+\\
&\epsilon_{ijk}(f_i\lambda_j\psi_k+\frac{1}{2}g_i(\lambda_j\lambda_k-\psi_j\psi_k))]+\\
&-\Phi_y[\dot{x}(\psi_0\lambda_0-\psi_i\lambda_i)+f_i(\psi_i\psi_0+\lambda_i\lambda_0)+g_i(\psi_i\lambda_0-\lambda_i\psi_0)-\\
&-\epsilon_{ijk}(g_i\psi_j\lambda_k-\frac{1}{2}f_i(\lambda_j\lambda_k-\psi_j\psi_k))]+\\
&+\Phi_{xx}[\frac{1}{6}\epsilon_{ijk}(\psi_i\psi_j\psi_k-3\lambda_i\lambda_j\psi_k)\psi_0]+\\
&+\Phi_{yy}[\frac{1}{6}\epsilon_{ijk}(\lambda_i\lambda_j\lambda_k-3\psi_i\psi_j\lambda_k)\lambda_0]-\\
&-\Phi_{xy}[\frac{1}{6}\epsilon_{ijk}(\psi_i\psi_j\psi_k\lambda_0+\lambda_i\lambda_j\lambda_k\psi_0+3\psi_i\lambda_j\lambda_k\lambda_0+3\lambda_i\psi_j\psi_k\psi_0)].
\end{aligned}$$

$\Phi(x,y)$ is a conformal factor (the induced metric on the $2D$ target is conformally flat).

Minimal representations:

1) "root" multiplets (n, n) induced by Clifford irreps

$$Q_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i \cdot H & 0 \end{pmatrix},$$

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \tilde{\sigma}_i & 0 \end{pmatrix} \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}.$$

2) higher-length multiplets induced by a dressing:

$$Q_i \mapsto \widehat{Q}_i = D Q_i D^{-1}$$

realized by a diagonal dressing matrix D.

Corollary: the total number of bosonic (fermionic) fields is given by

$$\begin{aligned} N &= 8l + m, \\ n &= 2^{4l} G(m), \end{aligned}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$. $G(m)$ is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

The non-minimal representations for $N \geq 4$ (reducible but indecomposable) are obtained from the "enveloping" representations

$$\mathbf{1}, \quad Q_i\mathbf{1}, \quad Q_iQ_j\mathbf{1}, \quad Q_iQ_jQ_k\mathbf{1}, \quad \ldots .$$

The field-content at mass-dimension $\frac{k}{2}$ is given by the Newton binomial.

Examples:
$N = 4$: $(1, 4, 6, 4, 1)$.
$N = 5$: $(1, 5, 10, 10, 5, 1)$
...

The enveloping representations can be dressed:

$$(1, 5, 10, 10, 5, 1) \rightarrow (0, 5, 11, 10, 5, 1).$$

The Schur lemma can be extended to minimal supermultiplet
(0, 1, 3 matrices τ_j commuting with the all the Γ's closing $\mathbf{1}, u(1), su(2)$ groups)
One has to check the compatibility with the dressing D ($[D, \tau_j] = 0$).

Example: Schur property of the $N = 5$ super-multiplets:

fields cont.	$N = 4$ decomp.	ψ_g connectivities	labels	group
$(8,8)$	$(4,4)+(4,4)$	8_0		$su(2)$
$(1,8,7)$	$(0,4,4)+(1,4,3)$	3_5+5_4		$-$
$(2,8,6)$	$(0,4,4)+(2,4,2)$	$2_5+2_4+4_3$	A	$u(1)$
	$(1,4,3)+(1,4,3)$	6_4+2_3	B	$-$
$(3,8,5)$	$(0,4,4)+(3,8,5)$	$1_5+3_4+4_2$	A	$-$
	$(1,4,3)+(2,4,2)$	$2_4+5_3+1_2$	B	$-$
$(4,8,4)$	$(0,4,4)+(4,4,0)$	4_4+4_1	A	$su(2)$
	$(1,4,3)+(3,4,1)$	$1_4+3_3+3_2+1_1$	B	$-$
	$(2,4,2)+(2,4,2)$	4_3+4_2	C	$u(1)$
$(5,8,3)$	$(1,4,3)+(4,4,0)$	$4_3+3_1+1_0$	A	$-$
	$(2,4,2)+(3,4,1)$	$1_3+5_2+2_1$	B	$-$
$(6,8,2)$	$(2,4,2)+(4,4,0)$	$4_2+2_1+2_0$	A	$u(1)$
	$(3,4,1)+(3,4,1)$	2_2+6_1	B	$-$
$(7,8,1)$	$(3,4,1)+(4,4,0)$	5_1+3_0		$-$
$(1,5,7,3)$	$(1,4,3)+(0,1,4,3)$	5_4		$-$
$(1,6,7,2)$	$(1,4,3)+(0,2,4,2)$	1_5+5_4		$-$
$(1,7,7,1)$	$(1,4,3)+(0,3,4,1)$	2_5+5_4		$-$
$(2,6,6,2)$	$(2,4,2)+(0,2,4,2)$	2_4+4_3		$u(1)$
$(2,7,6,1)$	$(2,4,2)+(0,3,4,1)$	$1_5+2_4+4_3$		$-$
$(3,7,5,1)$	$(3,4,1)+(0,3,4,1)$	3_4+4_2		$-$

Comment: for the $(8,8)$ root multiplet,

$N = 5$ generators are $su(2)$-invariant.

$N = 6$ generators are $u(1)$-invariant.

No invariance for $N = 7, 8$ generators.

Pre-Hopf map (bosonic):

for $k = 1, 2, 4, 8$

$$p \;:\; \mathbf{R}^{2k} \to \mathbf{R}^{k+1}.$$

It is a bilinear map

$$x_\mu \;=\; u^T \Gamma_\mu u$$

The norm is preserved by the mapping.
Therefore the restriction r

$$r \;:\; \mathbf{R}^{2k} \to \mathbf{S}^{2k-1},$$

induces the Hopf map h:

$$h \;:\; \mathbf{S}^{2k-1} \to \mathbf{S}^{k}.$$

Comment: we have 4 spaces (I, II, III, IV) and their mutual maps.

The spheres can be parametrized by the stereographic projection.

$1st$ Hopf map

$(I = \mathbf{R}^4,\ II = \mathbf{R}^{\mathbf{3}},\ III = \mathbf{S}^{\mathbf{3}},\ IV = \mathbf{S}^{\mathbf{2}})$.

$2nd$ Hopf map

$(I = \mathbf{R}^8,\ II = \mathbf{R}^{\mathbf{5}},\ III = \mathbf{S}^{\mathbf{7}},\ IV = \mathbf{S}^4)$.

$3rd$ Hopf map

$(I = \mathbf{R}^{16},\ II = \mathbf{R}^{\mathbf{9}},\ III = \mathbf{S}^{\mathbf{15}},\ IV = \mathbf{S}^{\mathbf{8}})$.

Supersymmetric extensions. Consider a root multiplet in I:

$1st$ Hopf map

$N = 3, 4$ and $(4, 4)$.

$2nd$ Hopf map

$N = 5, 6, 7, 8$ and $(8, 8)$.

$3rd$ Hopf map

$N = 9$ and $(16, 16)$.

Induced supermultiplets in II, III, IV.

1st Hopf map:

$N = 4$ $(4,4)$ in I is $U(1)$-invariant.

It induces $N = 4$ multiplets:

a $U(1)$-invariant $(3,4,1)$ linear multiplet in II.

a NONLINEAR $(3,4,1)$ in III.

a NONLINEAR $(2,4,2)$ in IV.

The supersymmetric extension of the 1st Hopf map is the mapping

$$(3,4,1)_{NL} \;\rightarrow\; (2,4,2)_{NL}$$

It is a nonlinear dressing.

The nonlinearity in III and IV is mild: susy transforms with at most bilinear fields.

Fixing R, the radius of the spheres, in the contraction limit $R \rightarrow \infty$ we recover the linear $(3,4,1)$ and $(2,4,2)$ $N = 4$ multiplets.

$N = 4$ $1D$ sigma-models associated to the first Hopf map.

S_I, S_{II}, S_{III}, S_{IV} (in I, II, III, IV) depending on UNCONSTRAINED prepotentials F.

In I F is function of 4 target coordinates.

In II F is function of 3 target coordinates.

In III F is function of 3 target coordinates.

In IV F is function of 2 target coordinates.

In I, II the induced metric g_{ij} on the target space is conformally flat (the susy transforms are linear).

Due to the nonlinearity of the susy transformations, in III, IV the induced metric possesses a non-trivial curvature even for a quadratic prepotential.

Comment: we have maps mutually relating the 4 sigma-model actions.

Example. In IV $((2,4,2)_{NL})$, parametrized by the target coordinates z_1, z_2, if we set the prepotential F to be function of $\rho = \sqrt{{z_1}^2 + {z_2}^2}$, we obtain that for a quadratic prepotential ($F(\rho) = C\rho^2$), the curvature scalar R depends on ρ:

$$R(\rho) = \frac{-44 + \rho^2}{C(\rho^2+1)^2(\rho^2-8)^2}$$

($R = -2$ at the origin requires $C = \frac{11}{32}$).

An inverse problem can be formulated: find a prepotential which reproduces a given target metric. As an example, the UNIFORMIZING PREPOTENTIAL (the curvature is constant everywhere).

At least locally we can compute a uniformizing prepotential by Taylor expansion. Example. $F = C(\rho^2 + k\rho^4)$ gives a vanishing 1st, 2nd, 3rd derivative of R at the origin for

$$\begin{aligned} C &\approx 51.27994 \\ k &\approx -1.97997. \end{aligned}$$

Supersymmetric extension of the $2nd$ Hopf map. The pre-Hopf supersymmetrization $I \to II$.

In I, several possibilities:

a) $N = 8$ $(8, 8)$,
b) $N = 6$ $(8, 8)$ with $U(1)$-invariance,
c) $N = 5$ $(8, 8)$ with $SU(2)$ invariance,
d) $(8, 8)$ with $N = 4 + BRST$ and $SU(2)$-invariance.

a) induces in III a HUGE linear multiplet with 128 bosonic and 128 fields.

c) induces in III the linear multiplet $(5, 11, 10, 5, 1)$, whose fields are bilinear composites of the fields in I and $SU(2)$-invariant.

What about the c) invariant action in II?
It admits a manifest $N = 4$ $SU(2)$-invariant lagrangian

$$L = Q_1 Q_2 Q_3 Q_4 F$$

$N = 5$ requires a Q_5 constraint on F:

$$Q_5 L \;=\; \partial_t R.$$

In this case the metric of the $5D$ target is conformally flat. The conformal factor Φ satisfies the Laplace equation $\nabla\Phi = 0$ when expressed in terms of the 8 target coordinates of I.

Comment: for $\nabla F = 0$ the action is both $N = 8$-invariant and $SU(2)$-invariant! (remember $N = 5 \rightarrow N = 8$).

The action contains only the fields of the $N = 4$ $(5, 8, 3)$ non-minimal multiplet associated to the manifest 4 supersymmetries.

If F is chosen to depend only on

$$\rho \;=\; \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2},$$

the action is $SO(5)$-invariant.

$N = 4$ supersymmetric mechanics with Yang monopole (Krivonos-Neressian-FT, in preparation).

Comment: $N = 4$ invariance, $SU(2)$-invariance and $SO(5)$-invariance ($N = 5$ is NOT imposed).

The simplest choice of prepotential is quadratic in the fields of I (an $(8, 8)$ root-multiplet decomposed, for $N = 4$, in two independent $N = 4$ $(4, 4)$ multiplets). The corresponding action has free constant kinetic term.

Grateful for Your kind attention.

This is the end.